GENERAL FORMS OF THE MENSHOV–RADEMACHER, ORLICZ, AND TANDORI THEOREMS ON ORTHOGONAL SERIES

VLADIMIR A. MIKHAILETS, ALEKSANDR A. MURACH

Abstract. We prove that the classical Menshov–Rademacher, Orlicz, and Tandori theorems remain true for orthogonal series given in the direct integrals of measurable collections of Hilbert spaces. In particular, these theorems are true for the spaces $L_2(X, d\mu; H)$ of vector-valued functions, where (X, μ) is an arbitrary measure space, and H is a real or complex Hilbert space of an arbitrary dimension.

1. Introduction

The Menshov–Rademacher theorem [1, 2] plays an important role in the theory of orthogonal series. It states that the sequence $(\log_2^2 n)$ is a Weyl multiplier for convergence, almost everywhere (a.e.) with respect to the Lebesgue measure, of a series in an arbitrary orthonormal system (ONS) of real-valued functions given on a finite interval of the real axis. There are some various theorems on unconditional convergence of orthogonal series. These results refine the Menshov–Rademacher theorem (see, e.g., [3, Ch. 2, § 5] and [4, Ch. 8, § 2]), where the Orlicz theorem [5] occupies a special place. It gives a sufficient condition for the sequence $(\omega_n \log_2^2 n)$ to be a Weyl multiplier for the unconditional convergence a.e. The Menshov–Rademacher and the Orlicz theorems are best possible in the sense that their conditions cannot be weakened.

It is known (see, e.g., [6, 7]) that the Menshov–Rademacher theorem remains valid for series with respect to ONSs of real-valued or complex-value functions given on an arbitrary measure space. This also true [8] for the Orlicz theorem and for another known result on unconditional convergence, the Tandori theorem [9].

The question arises whether these and others theorems on convergence of orthogonal series are true in a more general setting of series with respect to ONSs of vector-valued functions given on a measure space and taking values in a collection of Hilbert spaces.

In the present paper, we will give a positive answer to this question for the classical Menshov–Rademacher, Orlicz, and Tandori theorems.

Note that, in the case of orthogonal series in (complex-valued) eigenfunctions of a self-adjoint elliptic operator defined on a closed compact manifold X, the conditions of the Menshov–Rademacher and the Orlicz theorems and that the function being expanded belongs to the isotropic Hörmander spaces $H^{\psi}(X)$ are equivalent, where $\psi(t) = \log^* t$ or $\psi(t) = \varphi(t) \log^* t$, respectively; see [10, 11] and [12, Sec. 2.3.2]. Here $\log^* t := \max\{1, \log_2 t\}$, whereas $\varphi(t)$, $t \geq 1$, is a positive increasing function that varies regularly at $+\infty$ in the sense

2000 *Mathematics Subject Classification.* Primary 40A30; Secondary 46E40.

Key words and phrases. Orthogonal series, almost everywhere convergence, unconditional convergence, Menshov–Rademacher theorem, Orlicz theorem, Tandori theorem, direct integral of Hilbert spaces.

of Karamata and satisfies the condition

$$\int_{2}^{\infty} \frac{dt}{t\,(\log_2 t)\,\varphi^2(t)} < \infty.$$

2. Statements of the main results

Let X be an arbitrary measurable space with some σ-additive measure $\mu \geq 0$. The measure is not assumed to be finite or σ-finite. Let $\{H(x) : x \in X\}$ be a μ-measurable collection of either real or complex Hilbert spaces. This means that the function $\dim H(x)$, $x \in X$, takes only finitely or countably many values (that are cardinal numbers) and that all the sets

$$\{\, x \in X :\ \dim H(x) = \mathrm{const}\,\}$$

are μ-measurable. We consider the direct integral

$$\mathbf{L}_2 := \int_X^{\oplus} H(x)\, d\mu(x)$$

of the μ-measurable collection $\{H(x) : x \in X\}$ (see, e.g., [13, Ch. 7, Sec. 1] and [14, Ch. 2]). The space $\mathbf{L}_2$ is endowed with the inner product

$$(f(\cdot), g(\cdot))_2 := \int_X (f(x), g(x))_{H(x)}\, d\mu(x),$$

which induces the norm $\|\cdot\|_2$.

If $H(x) \equiv H = \mathrm{const}$, then

$$\mathbf{L}_2 = L_2(X, d\mu; H) = L_2(X, d\mu) \otimes H.$$

Thus, in this case, the space $\mathbf{L}_2$ consists of all classes of μ-equivalent vector-valued functions $f : X \to H$ that are strongly measurable with respect to μ [15, Ch. V, Sec. 4] and that

$$\|f\|_2 = \biggl(\int_X \|f(x)\|_H^2\, d\mu(x)\biggr)^{1/2} < \infty.$$

Let an ONS of vector-valued functions $\Phi := (\varphi_n)_{n=1}^{\infty}$ be arbitrarily chosen in the space $\mathbf{L}_2$. We investigate the μ-almost everywhere (μ-a.e.) convergence on X of the orthogonal series

$$\sum_{n=1}^{\infty} a_n\, \varphi_n(x). \tag{1}$$

Here all coefficients a_n are either complex or real numbers; this depends on whether all the spaces $H(x)$, $x \in X$ are complex or real. We set $a := (a_n)_{n=1}^{\infty}$. Given $x \in X$, the convergence of the series (1) is regarded in the norm of $H(x)$.

Consider the majorant of partial sums of this series:

$$S^{*}(\Phi, a, x) := \sup_{m \in \mathbb{N}} \Bigl\| \sum_{n=1}^{m} a_n\, \varphi_n(x) \Bigr\|_{H(x)}, \quad x \in X. \tag{2}$$

Let us formulate the main results of the paper.

Theorem 1 (a general form of the Menshov–Rademacher theorem)**.** *Let a sequence of numbers* $(a_n)_{n=1}^{\infty}$ *satisfy the condition*

$$L := \sum_{n=1}^{\infty} |a_n|^2 \, \log_2^2(n+1) < \infty. \tag{3}$$

Then the series (1) *converges* μ*-a.e. on* X*, and moreover*

$$\|S^*(\Phi, a, \cdot)\|_2 \le K \sqrt{L}. \tag{4}$$

Here K *is a certain universal positive constant, one may take* $K = 4$*.*

This theorem was proved independently by D. E. Menshov [1] and H. Rademacher [2] in the case where

$$X = (\alpha, \beta), \quad -\infty < \alpha < \beta < \infty, \quad \mu \text{ is the Lebesgue measure}, \quad H(x) \equiv \mathbb{R}. \tag{5}$$

An exposition of their results are given, e.g., in G. Alexits' [3, Sec. 2.3.2] and B. S. Kashin and A. A. Saakyan's [4, Ch. 8, § 1] books. Note that the measures μ that are absolutely continuous with respect to the Lebesgue measure are also allowed in [3]. As it has been mentioned, the Menshov–Rademacher theorem remains true for the ONSs of real-valued or complex-valued functions given on an arbitrary measure space X. Remark that a complete characterization of the sequences $(a_n)_{n=1}^{\infty}$ such that the series (1) converges a.e. for an arbitrary ONS in $L_2(X, d\mu; \mathbb{R})$ is given by A. Paszkiewicz [16].

The Menshov–Rademacher theorem is precise. In the situation (5), D. E. Menshov [1] constructed an example of ONS $(\varphi_n)_{n=1}^{\infty}$ such that for every sequence of numbers $(\omega_n)_{n=1}^{\infty}$ satisfying

$$1 = \omega_1 \le \omega_2 \le \omega_3 \le \dots, \quad \lim_{n\to\infty} \frac{\omega_n}{\log_2^2 n} = 0$$

there exists an a.e. divergent series of the form (1) whose coefficients meet the condition

$$\sum_{n=1}^{\infty} |a_n|^2 \, \omega_n < \infty.$$

This result is presented, e.g., in the books [3, Sec. 2.4.1] and [4, Ch. 8, § 1] mentioned above.

Recall that the series (1) is called *unconditionally* convergent μ-a.e. on X if the series

$$\sum_{n=1}^{\infty} a_{\sigma(n)} \, \varphi_{\sigma(n)}(x) \tag{6}$$

converges μ-a.e. on X for an arbitrary permutation $\sigma = (\sigma(n))_{n=1}^{\infty}$ of the set $\mathbb{N}$ of all positive integers. Here the zero measure set of the points at which the series (6) diverges can depend on the permutation σ.

Theorem 2 (a general form of the Tandori theorem)**.** *Let a sequence of numbers* $(a_n)_{n=1}^{\infty}$ *satisfy the condition*

$$\sum_{k=0}^{\infty} \Bigg(\sum_{n=\nu_k+1}^{\nu_{k+1}} |a_n|^2 \, \log_2^2 n \Bigg)^{1/2} < \infty, \tag{7}$$

where $\nu_k := 2^{2^k}$*. Then the series* (1) *converges unconditionally* μ*-a.e. on* X*.*

This theorem was proved by K. Tandori [9] in the situation (5). He also showed that his theorem is best possible in the following sense. Given a (nonstrictly) decreasing sequence of positives numbers $(a_n)_{n=1}^{\infty}$, the series (1) converges unconditionally a.e. for each ONS $(\varphi_n)_{n=1}^{\infty}$ in $L_2((0;1),dx,\mathbb{R})$ if and only if (7) holds. These K. Tandori's results are presented in the book [4] (see Ch. 8, § 2 and the remarks to Ch. 8).

A sufficient condition for the unconditional convergence of the series (1) can be expressed in the terms of the Weyl multipliers.

Theorem 3 (a general form of the Orlicz theorem)**.** *Let a sequence of numbers $(a_n)_{n=1}^{\infty}$ and a (nonstrictly) increasing sequence of positives numbers $(\omega_n)_{n=1}^{\infty}$ satisfy the following conditions:*

$$\sum_{n=2}^{\infty} |a_n|^2\,(\log_2^2 n)\,\omega_n < \infty, \tag{8}$$

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\log_2 n)\,\omega_n} < \infty. \tag{9}$$

Then the series (1) *converges unconditionally μ-a.e. on X.*

Under the assumption (5), Theorem 3 is an equivalent formulation of the Orlicz theorem [5], which was suggested by P. L. Ulj'anov [17, § 4, Sec. 1] (also see [18, § 9, Sec. 1]). The Orlicz theorem and its proof can be founded, e.g., in G. Alexits' book [3, Sec. 2.5.1]. As K. Tandori proved [9], this theorem is best possible in the sense that the condition (9) on the sequence $(\omega_n)_{n=1}^{\infty}$ cannot be weakened.

Note that both Theorems 2 and 3 remain true for each ONS of complex-valued functions given on an arbitrary measure space X [8].

Theorems 1, 2, and 3 will be proved in Sections 4, 5, and 6, resp. When proving Theorems 1 and 2, we use the classical scheme of argumentation set forth in [4, Ch. 8, §1, 2] for the case (5). Theorem 3 will be deduced from Theorem 2. Previously, in Section 3 we establish a general form of the Menshov–Rademacher inequality that plays a decisive role in the proofs of Theorems 1 and 2.

3. Menshov-Rademacher inequality

The proofs of Theorem 1 and 2 are based on the following fact.

Lemma 1. *Let an integer $N \geq 1$, finite ONS of vector-valued functions $\Psi := (\psi_n)_{n=1}^{N}$ in $\mathbf{L}_2$, and a finite collection of numbers $b := (b_n)_{n=1}^{N}$ be given arbitrarily. Then the function*

$$S_N^{*}(\Psi,b,x) := \max_{1\leq j\leq N} \Big\| \sum_{n=1}^{j} b_n\,\psi_n(x)\,\Big\|_{H(x)}, \quad x \in X, \tag{10}$$

satisfies the inequality

$$\|S_N^{*}(\Psi,b,\cdot)\|_2 \leq (2+\log_2 N)\left(\sum_{n=1}^{N} |b_n|^2\right)^{1/2}. \tag{11}$$

In the classical case (5), the inequality (11) was obtained independently by D. E. Menshov [1] and G. Rademacher [2] and then used by them in the proof of Theorem 1 (see, e.g., the books [3, Sec. 2.3.1, 2.3.2] and [4, Ch. 9, § 1]). On the right-hand side of (11), the factor $C\log_2(N+1)$ with some universal constant C is used usually instead of $2+\log_2 N$. Note

that this inequality is known for ONSs of real-valued or complex-valued functions given on an arbitrary measure space X (see, e.g., [19, Theorem 3] and [7, Proposition 2.1]).

Proof of Lemma 1. First we consider the case when $N = 2^r$ for some integer $r \ge 1$. The general situation is easily reduced to this case; this will be shown at the end of the proof.

Given an arbitrary number $j \in \{1, 2, \dots, 2^r\}$, consider its binary representation

$$j = \sum_{k=0}^{r} \varepsilon_k \, 2^{r-k}, \quad \text{where } \varepsilon_k := \varepsilon_k(j) \in \{0, 1\}.$$

Then every sum $\sum_{n=1}^{j} h_n$ of vectors in a real or complex Hilbert space H can be represented in the form

$$\sum_{n=1}^{j} h_n = \sum_{k\,:\,\varepsilon_k \neq 0} \quad \sum_{\sum_{s=0}^{k-1} \varepsilon_s 2^{r-s} < n \le \sum_{s=0}^{k} \varepsilon_s 2^{r-s}} h_n.$$

Whence, using the triangle inequality for the norm in H and the Cauchy inequality (both being applied to the external sum of $\le r+1$ terms), we get:

$$\begin{aligned}
\Big\| \sum_{n=1}^{j} h_n \Big\|_H &= \Big\| \sum_{k\,:\,\varepsilon_k \neq 0} 1 \cdot \sum_{\sum_{s=0}^{k-1} \varepsilon_s 2^{r-s} < n \le \sum_{s=0}^{k} \varepsilon_s 2^{r-s}} h_n \Big\|_H \\
&\le \sum_{k\,:\,\varepsilon_k \neq 0} 1 \cdot \Big\| \sum_{\sum_{s=0}^{k-1} \varepsilon_s 2^{r-s} < n \le \sum_{s=0}^{k} \varepsilon_s 2^{r-s}} h_n \Big\|_H \\
&\le (r+1)^{1/2} \Bigg(\sum_{k\,:\,\varepsilon_k \neq 0} \Big\| \sum_{\sum_{s=0}^{k-1} \varepsilon_s 2^{r-s} < n \le \sum_{s=0}^{k} \varepsilon_s 2^{r-s}} h_n \Big\|_H^2 \Bigg)^{1/2} \\
&\le (r+1)^{1/2} \Bigg(\sum_{k=0}^{r} \sum_{p=0}^{2^k-1} \Big\| \sum_{n=p2^{r-k}+1}^{(p+1)2^{r-k}} h_n \Big\|_H^2 \Bigg)^{1/2}.
\end{aligned}$$

Thus

$$\Big\| \sum_{n=1}^{j} h_n \Big\|_H^2 \le (r+1) \sum_{k=0}^{r} \sum_{p=0}^{2^k-1} \Big\| \sum_{n=p2^{r-k}+1}^{(p+1)2^{r-k}} h_n \Big\|_H^2. \tag{12}$$

We apply this inequality to estimate the function (10), which is represented in the form

$$S_N^*(\Psi, b, x) = \Big\| \sum_{n=1}^{j(x)} b_n \, \psi_n(x) \Big\|_{H(x)}, \quad x \in X;$$

here the number $j(x) \in \{1, 2, \dots, 2^r\}$ is properly chosen for every fixed $x \in X$. Setting $h_n := b_n \psi_n(x)$ in (12), write:

$$(S_N^*(\Psi, b, x))^2 \le (r+1) \sum_{k=0}^{r} \sum_{p=0}^{2^k-1} \Big\| \sum_{n=p2^{r-k}+1}^{(p+1)2^{r-k}} b_n \psi_n(x) \Big\|_{H(x)}^2, \quad x \in X.$$

Integrating the latter inequality and using that $(\psi_n)_{n=1}^{2^r}$ is an ONS in $\mathbf{L}_2$, we have:

$$\|S_N^*(\Psi,b,\cdot)\|_2^2 \leq (r+1)\sum_{k=0}^{r}\sum_{p=0}^{2^k-1}\int\limits_X \Big\|\sum_{n=p2^{r-k}+1}^{(p+1)2^{r-k}} b_n\psi_n(x)\Big\|_{H(x)}^2 d\mu(x)$$

$$= (r+1)\sum_{k=0}^{r}\sum_{p=0}^{2^k-1}\sum_{n=p2^{r-k}+1}^{(p+1)2^{r-k}} |b_n|^2 = (r+1)\sum_{k=0}^{r}\sum_{n=1}^{2^r}|b_n|^2 = (r+1)^2\sum_{n=1}^{2^r}|b_n|^2.$$

Thus

$$\|S_N^*(\Psi,b,\cdot)\|_2^2 \leq (r+1)^2\sum_{n=1}^{2^r}|b_n|^2. \tag{13}$$

This, in view of $N=2^r$, yields the required estimate (11).

Now consider the general situation, when $N\geq 1$ is an arbitrary integer. If $N=1$, then Lemma 1 is trivial. Let $N\geq 2$; then there exists an integer $r\geq 1$ such that $2^{r-1}<N\leq 2^r$. Putting $a_n:=0$ for $N<n\leq 2^r$, we arrive at the above case, when the collection (a_n) consists of 2^r numbers. Therefore, (13) holds with $r-1<\log_2 N$; i.e., the required inequality (11) is fulfilled in the general situation.

Lemma 1 is proved.

4. Proof of Theorem 1

Beforehand let us make a useful remark. Without loss of generality we may assume that the measure μ is σ-finite. Indeed, since $\|\varphi_n\|_2=1$ for each $n\geq 1$, it follows that every set $\{x\in X : \|\varphi_n(x)\|_{H(x)}>1/j\}$, with $j\in\mathbb{N}$, has a finite measure. Hence, μ is a σ-finite measure on the set of all points $x\in X$ such that $\varphi_n(x)\neq 0$ for at least one index n. Outside this set all terms of the series (1) are zero-vectors. Therefore our assumption does not lead to any loss of generality in the proofs.

Now let us show that the sequence

$$S_{2^k}(x) := \sum_{n=1}^{2^k} a_n\,\varphi_n(x), \quad k=1,\,2,\,3,\ldots, \tag{14}$$

converges for μ-a.e. $x\in X$, and then we estimate the norm in $L_2(X,d\mu;\mathbb{R})$ of the function

$$S^\star(x) := \sup_{0\leq k<\infty}\|S_{2^k}(x)\|_{H(x)}, \quad x\in X.$$

Let

$$\chi_k(x) := \sum_{n=2^k}^{2^{k+1}-1} a_n\,\varphi_n(x), \quad x\in X, \quad k=0,\,1,\,2,\,3,\ldots.$$

Since $(\varphi_n)_{n=1}^{\infty}$ is an ONS in $\mathbf{L}_2$, we may write

$$\|\chi_k\|_2^2 = \sum_{n=2^k}^{2^{k+1}-1}|a_n|^2.$$

Hence, by the condition (3), we have

$$\sum_{k=0}^{\infty} \|\chi_k\|_2^2 \, (k+1)^2 = \sum_{k=0}^{\infty} (k+1)^2 \sum_{n=2^k}^{2^{k+1}-1} |a_n|^2$$
$$\leq \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} |a_n|^2 \, (1+\log_2 n)^2 \leq 2L < \infty.$$

Whence, applying the Cauchy inequality, we get

$$\sum_{k=0}^{\infty} \|\chi_k\|_2 = \sum_{k=0}^{\infty} \|\chi_k\|_2 \, (k+1) \, (k+1)^{-1}$$
$$\leq \left(\sum_{k=0}^{\infty} \|\chi_k\|_2^2 \, (k+1)^2 \right)^{1/2} \left(\sum_{k=0}^{\infty} (k+1)^{-2} \right)^{1/2} \leq \sqrt{2L}\,\sqrt{2} = 2\sqrt{L}.$$

Thus

$$\sum_{k=0}^{\infty} \|\chi_k\|_2 \leq 2\sqrt{L}. \tag{15}$$

Let us show that

$$\sum_{k=0}^{\infty} \|\chi_k(x)\|_{H(x)} < \infty \quad \text{for} \quad \mu\text{-a.e } \ x \in X. \tag{16}$$

Recall that without loss of generality we may consider measure μ to be σ-finite on X.

If $\mu(X) < \infty$, then by (15) and the Cauchy inequality we have:

$$\begin{aligned} \sum_{k=0}^{\infty} \int_X \|\chi_k(x)\|_{H(x)} \, d\mu(x) &\leq \sum_{k=0}^{\infty} \left(\int_X d\mu(x) \right)^{1/2} \left(\int_X \|\chi_k(x)\|_{H(x)}^2 \, d\mu(x) \right)^{1/2} \\ &\leq 2\sqrt{\mu(X)\,L} < \infty. \end{aligned} \tag{17}$$

Therefore, according to the B. Levi theorem, we may write

$$\int_X \left(\sum_{k=0}^{\infty} \|\chi_k(x)\|_{H(x)} \right) d\mu(x) = \sum_{k=0}^{\infty} \int_X \|\chi_k(x)\|_{H(x)} \, d\mu(x) < \infty; \tag{18}$$

this yields (16).

If $\mu(X) = \infty$, then represent X as a countable union of some measurable sets X_j, $j = 1, 2, 3, \ldots$, with $\mu(X_j) < \infty$. For every j formula (17) and its consequences, formulas (18) and (16), remain true if we replace X by X_j. So, we get (16) again.

It follows from (16) that (14) is a Cauchy sequence for μ-a.e. $x \in X$, i.e., (14) converges. Besides,

$$S^\star(x) \leq \sum_{k=0}^{\infty} \|\chi_k(x)\|_{H(x)} < \infty \quad \text{for} \ \ \mu\text{-a.e.} \ \ x \in X.$$

Whence we have by (15) that:

$$\|S^\star\|_2 \leq \sum_{k=0}^{\infty} \|\chi_k\|_2 \leq 2\sqrt{L}. \tag{19}$$

Now consider the function

$$S^{\circ}(x) := \sup_{1\leq k<\infty} S_k^{\circ}(x), \quad x \in X,$$

where

$$S_k^{\circ}(x) := \max_{2^k\leq j<2^{k+1}} \Big\| \sum_{n=2^k}^{j} a_n\,\varphi_n(x) \Big\|_{H(x)}, \quad x \in X, \quad k = 1,\,2,\,3,\ldots .$$

Applying Lemma 1, with $\Psi := (\varphi_n)_{n=2^k}^{j}$ and $b := (a_n)_{n=2^k}^{j}$, and using the condition (3), we may write the following:

$$\begin{aligned}\sum_{k=1}^{\infty} \|S_k^{\circ}\|_2^2 &\leq \sum_{k=1}^{\infty} \max_{2^k\leq j<2^{k+1}} \big(2+\log_2(j-2^k+1)\big)^2 \sum_{n=2^k}^{j} |a_n|^2 \\ &\leq \sum_{k=1}^{\infty} \big(2+\log_2 2^k\big)^2 \sum_{n=2^k}^{2^{k+1}-1} |a_n|^2 \leq \sum_{k=1}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} |a_n|^2\,(2+\log_2 n)^2 \\ &= \sum_{n=1}^{\infty} |a_n|^2\,(2+\log_2 n)^2 \leq 4L < \infty.\end{aligned}$$

Therefore, by the B. Levy theorem, we have

$$\int_X \Big(\sum_{k=1}^{\infty} (S_k^{\circ}(x))^2 \Big)\, d\mu(x) = \sum_{k=1}^{\infty} \int_X (S_k^{\circ}(x))^2\, d\mu(x) \leq 4L < \infty. \tag{20}$$

Whence $\lim_{k\to\infty} S_k^{\circ}(x) = 0$ for μ-a.e. $x \in X$. This together with the convergence of (14) for μ-a.e. $x \in X$ proved above yields the convergence of the sequence (3) for μ-a.e. $x \in X$.

Moreover, since

$$S^{*}(\Phi,a,x) \leq S^{\star}(x) + S^{\circ}(x), \quad \big(S^{\circ}(x)\big)^2 \leq \sum_{k=1}^{\infty} \big(S_k^{\circ}(x)\big)^2, \quad x \in X,$$

we finally deduce the required inequality (4) from (19) and (20),

$$\|S^{*}(\Phi,a,\cdot)\|_2 \leq \|S^{\star}\|_2 + \|S^{\circ}\|_2 \leq 4\sqrt{L}.$$

Theorem 1 is proved.

5. Proof of Theorem 2

Without loss of generality we may assume that $a_1 = a_2 = 0$. Denote for an integer $k \geq 0$:

$$M_k := \{ j \in \mathbb{N} : \, \nu_k + 1 \leq j \leq \nu_{k+1} \};$$

recall that $\nu_k := 2^{2^k}$. Consider an arbitrary permutation (6) of the orthogonal series (1). Define a sequence of numbers $\big(\varepsilon_n^{(k)}\big)_{n=1}^{\infty}$ by the formula

$$\varepsilon_n^{(k)} := \begin{cases} 1, & \text{if} \quad \sigma(n) \in M_k, \\ 0, & \text{otherwise.} \end{cases}$$

Given arbitrary $p, q \in \mathbb{N}$ with $p \le q$, we may write

$$\sum_{n=p}^{q} a_{\sigma(n)}\,\varphi_{\sigma(n)}(x) = \sum_{k=0}^{\infty}\sum_{n=p}^{q} \varepsilon_n^{(k)}\,a_{\sigma(n)}\,\varphi_{\sigma(n)}(x), \quad x \in X. \tag{21}$$

The series on the right of (21) converges for every $x \in X$ because it contains only a finitely many of nonzero terms.

Given any integer $k \ge 0$, we set

$$\delta_k(x) := \sup_{1\le p<q<\infty} \Big\| \sum_{n=p}^{q} \varepsilon_n^{(k)}\,a_{\sigma(n)}\,\varphi_{\sigma(n)}(x) \Big\|_{H(x)}, \quad x \in X. \tag{22}$$

Note that

$$\delta_k(x) \;\le\; 2 \sup_{1\le q<\infty} \Big\| \sum_{n=1}^{q} \varepsilon_n^{(k)}\,a_{\sigma(n)}\,\varphi_{\sigma(n)}(x) \Big\|_{H(x)}, \quad x \in X; \tag{23}$$

here the sum contains only the terms with $\sigma(n) \in M_k$. We put in, Lemma 1,

$$\begin{aligned} \Psi &:= \{\varphi_{\sigma(n)} : \, n \in \mathbb{N} \text{ such that } \sigma(n) \in M_k\},\\ b &:= \{a_{\sigma(n)} : \, n \in \mathbb{N} \text{ such that } \sigma(n) \in M_k\},\\ N = N(k) &:= \nu_{k+1} - \nu_k = \nu_k(\nu_k - 1). \end{aligned}$$

Then

$$S^*_{N(k)}(\Psi, b, x) = \sup_{1\le q<\infty} \Big\| \sum_{n=1}^{q} \varepsilon_n^{(k)}\,a_{\sigma(n)}\,\varphi_{\sigma(n)}(x) \Big\|_{H(x)}, \quad x \in X.$$

Therefore, by Lemma 1 and in view of (23), we have

$$\begin{aligned} \|\delta_k\|_2 &\le (4 + 2\log_2 N(k)) \Big(\sum_{n\,:\,\sigma(n)\in M_k} |a_{\sigma(n)}|^2 \Big)^{1/2}\\ &= (4 + 2\log_2 N(k)) \Big(\sum_{n=\nu_k+1}^{\nu_{k+1}} |a_n|^2 \Big)^{1/2}. \end{aligned}$$

Hence, since

$$4 + 2\log_2 N(k) = 4 + 2\log_2(\nu_k(\nu_k - 1)) \le 8\log_2 \nu_k,$$

we arrive at the estimate

$$\Big(\int_X \delta_k^2(x)\,d\mu(x) \Big)^{1/2} \le 8 \Big(\sum_{n=\nu_k+1}^{\nu_{k+1}} |a_n|^2 \log_2^2 n \Big)^{1/2}. \tag{24}$$

We will deduce from this that

$$\sum_{k=0}^{\infty} \delta_k(x) < \infty \quad \text{для} \quad \mu\text{-п.в. } x \in X. \tag{25}$$

Recall that, without loss of generality, the measure μ is assumed to be σ-finite on X.

If $\mu(X)<\infty$, then by the Cauchy inequality for integrals, the estimate (24), and condition (7) we may write the following:

$$\begin{aligned}\sum_{k=0}^{\infty}\int\limits_X \delta_k(x)\,d\mu(x) &\leq \sum_{k=0}^{\infty}\Big(\int\limits_X d\mu(x)\Big)^{1/2}\Big(\int\limits_X \delta_k^2(x)\,d\mu(x)\Big)^{1/2}\\ &\leq 8\sqrt{\mu(X)}\,\sum_{k=0}^{\infty}\Big(\sum_{n=\nu_k+1}^{\nu_{k+1}}|a_n|^2\,\log_2^2 n\Big)^{1/2}<\infty.\end{aligned}\tag{26}$$

Therefore, according to the B. Levi theorem, we have

$$\int\limits_X\Big(\sum_{k=0}^{\infty}\delta_k(x)\Big)\,d\mu(x)=\sum_{k=0}^{\infty}\int\limits_X\delta_k(x)\,d\mu(x)<\infty,\tag{27}$$

whence we get (25) (recall that all $\delta_k\geq 0$).

If $\mu(X)=\infty$, then represent X as a countable union of measurable sets X_j, $j\in\mathbb{N}$, with $\mu(X_j)<\infty$. For every j the inequality (26) and its consequences, formulas (27) and (25), remains valid if we replace X by X_j. Whence we obtain (25) again.

By (25), for μ-a.e. $x\in X$ and arbitrary $\varepsilon>0$ there exists a number $m=m(x,\varepsilon)$ such that

$$\sum_{k=m}^{\infty}\delta_k(x)<\varepsilon.\tag{28}$$

Let $p=p(x,\varepsilon)$ be large enough so that the sum

$$\sum_{n=1}^{p-1}a_{\sigma(n)}\,\varphi_{\sigma(n)}(x)$$

contains all the functions φ_n whose indexes belong to M_k with $0\leq k<m(x,\varepsilon)$. Then by (22) and (28) we have for every $q\geq p$ that

$$\begin{aligned}\Big\|\sum_{n=p}^{q}a_{\sigma(n)}\,\varphi_{\sigma(n)}(x)\Big\|_{H(x)} &= \Big\|\sum_{k=0}^{\infty}\sum_{n=p}^{q}\varepsilon_n^{(k)}\,a_{\sigma(n)}\,\varphi_{\sigma(n)}(x)\Big\|_{H(x)}\\ &= \Big\|\sum_{k=m}^{\infty}\sum_{n=p}^{q}\varepsilon_n^{(k)}\,a_{\sigma(n)}\,\varphi_{\sigma(n)}(x)\Big\|_{H(x)}\\ &\leq \sum_{k=m}^{\infty}\Big\|\sum_{n=p}^{q}\varepsilon_n^{(k)}\,a_{\sigma(n)}\,\varphi_{\sigma(n)}(x)\Big\|_{H(x)}\leq \sum_{k=m}^{\infty}\delta_k(x)<\varepsilon.\end{aligned}$$

Thus, for μ-a.e. $x\in X$ and for an arbitrary $\varepsilon>0$ there exists a number $p=p(x,\varepsilon)$ such that

$$\Big\|\sum_{n=p}^{q}a_{\sigma(n)}\,\varphi_{\sigma(n)}(x)\Big\|_{H(x)}<\varepsilon$$

for every integer $q\geq p$. So, the series (6) converges for μ-a.e. $x\in X$.

Theorem 2 is proved.

6. Proof of Theorem 3

We deduce it from Theorem 2 by showing that the conditions (8) and (9) together imply (7).

For every integer $k \geq 0$, put

$$A_k := \sum_{n=\nu_k+1}^{\nu_{k+1}} |a_n|^2 \log_2^2 n;$$

here $\nu_k := 2^{2^k}$ as above. Applying the Cauchy inequality, we may write

$$\sum_{k=0}^{\infty} A_k^{1/2} = \sum_{k=0}^{\infty} A_k^{1/2} \omega_{\nu_k}^{1/2} \omega_{\nu_k}^{-1/2} \leq \left(\sum_{k=0}^{\infty} A_k \omega_{\nu_k} \right)^{1/2} \left(\sum_{k=0}^{\infty} \omega_{\nu_k}^{-1} \right)^{1/2}.$$

It is known that

$$\sum_{n=2}^{\infty} \frac{1}{n (\log_2 n) \omega_n} < \infty \;\Leftrightarrow\; \sum_{n=1}^{\infty} \frac{1}{n\, \omega_{2^n}} < \infty \;\Leftrightarrow\; c := \sum_{n=0}^{\infty} \frac{1}{\omega_{\nu_n}} < \infty.$$

Therefore, using (8) and since $(\omega_n)_{n=1}^{\infty}$ is increasing, we have the following:

$$\begin{aligned}\left(\sum_{k=0}^{\infty} A_k^{1/2} \right)^2 &\leq c \sum_{k=0}^{\infty} A_k \omega_{\nu_k} = c \sum_{k=0}^{\infty} \omega_{\nu_k} \sum_{n=\nu_k+1}^{\nu_{k+1}} |a_n|^2 \log_2^2 \\ &\leq c \sum_{k=0}^{\infty} \sum_{n=\nu_k+1}^{\nu_{k+1}} |a_n|^2 (\log_2^2 n)\, \omega_n = c \sum_{n=3}^{\infty} |a_n|^2 (\log_2^2 n)\, \omega_n < \infty.\end{aligned}$$

Thus, the condition (7) is satisfied,

$$\sum_{k=0}^{\infty} \left(\sum_{n=\nu_k+1}^{\nu_{k+1}} |a_n|^2 \log_2^2 n \right)^{1/2} = \sum_{k=0}^{\infty} A_k^{1/2} < \infty.$$

Therefore, by Theorem 2, the sequence (1) converges unconditionally μ-a.e. on X.

Theorem 3 is proved.

7. Final remark

A simple inspection of the proofs of Lemma 1 and Theorems 1–3 reveals that they remain true if the system $(\varphi_n)_{n=1}^{\infty}$ forms a Riesz basis in the closure of its linear span in $\mathbf{L}_2$. In this case, the factor $C \log_2(N+1)$ should be used, instead of $2 + \log_2 N$, in the right-hand side of (11), the constant $C > 0$ as well as K in Theorem 1 depending on a choice of $(\varphi_n)_{n=1}^{\infty}$.

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Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereshchenkivs'ka, Kyiv, 01601, Ukraine

E-mail address: `mikhailets@imath.kiev.ua, murach@imath.kiev.ua`